EECS 16B Section 6B

W-2/24

Main Topic: Bandpass Filters & Bode Plots

Administrivia:

- HW 6 due Fri, 2/26
- Anonymous Feedback:
  bit.ly/maxwell-16B-feedback-sp21

Agenda:

- Resonance
  - Q1: Resonant Bandpass Filter
  - Bandpass vs. Resonant Bandpass
- Log plots, Bode Plot Approx.
  - Q2: Straight-Line Approx.
- Module 1 Recap

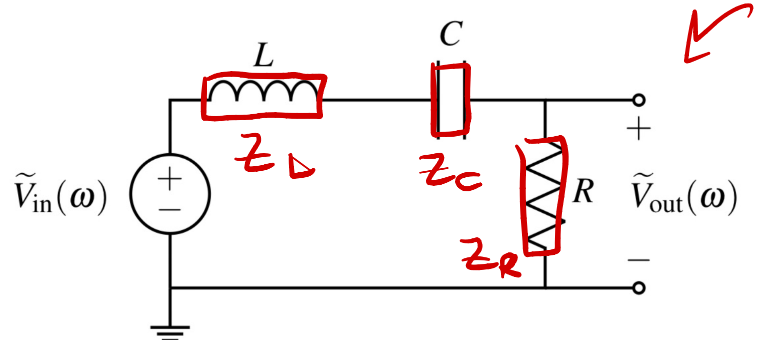
Resonance

- Phenomenon when a system experiences HIGH amplitude when at or near the natural frequency of the system.
- Analogies:
  - Opera singer breaks glass at certain pitch
  - "Sweet Spot" in tennis or golf
- Electrical resonance in LC circuits

1. Band-pass filter

It is quite common to need to design a filter which selects only a narrow range of frequencies. One example is in WiFi radios, it is desirable to select only the 2.4GHz frequency containing your data, and reject information from other nearby cellular or bluetooth frequencies. This type of filter is called a band-pass filter; we will explore the design of this type of filter in this problem.

"resonant"



(a) **Write down the impedance of the series RLC combination in the form $Z_{RLC}(\omega) = A(\omega) + jX(\omega)$, where $A(\omega)$ and $X(\omega)$ are real valued functions of ω.**

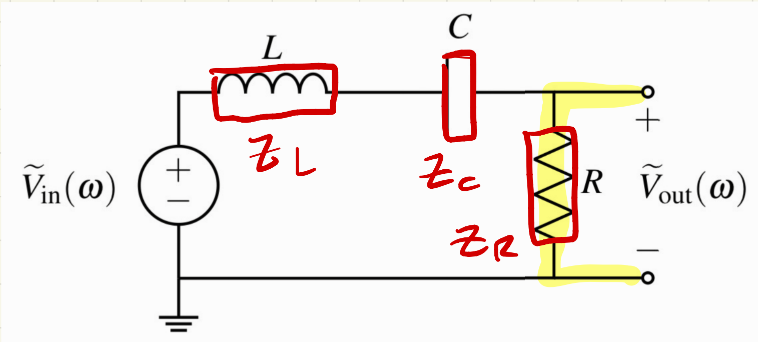
$$A = R$$
$$X = \omega L - \frac{1}{\omega C}$$

$$Z_{RLC} = Z_R + Z_L + Z_C$$
$$= R + j\omega L + \frac{1}{j\omega C}$$
$$= R + j\omega L - \frac{j}{\omega C}$$
$$= R + j\omega L - \frac{j}{\omega C}$$

$$\frac{1}{j}\left(\frac{j}{j}\right) = \frac{j}{j^2} = -j$$

$$\boxed{\begin{array}{l} A = R \\ X = \omega L - \frac{1}{\omega C} \end{array}}$$

(b) **Write down the transfer function $H(\omega) = \frac{\tilde{V}_{out}(\omega)}{\tilde{V}_{in}(\omega)}$ for this circuit.**



$$Z_{RLC} = R + j\omega L - \frac{j}{\omega C}$$

$$\tilde{V}_{out} = \frac{Z_R}{Z_{RLC}} \tilde{V}_{in}$$

$$\boxed{\frac{\tilde{V}_{out}}{\tilde{V}_{in}} = H(\omega) = \frac{R}{R + j\omega L - \frac{j}{\omega C}}}$$

(c) **At what frequency ω_n does $X(\omega_n) = 0$?** (i.e. at what frequency is the impedance of the series combination of RLC purely real — meaning that the imaginary terms coming from the capacitor and inductor completely cancel each other. This is called the *natural frequency*.)

What happens to the relative magnitude of the impedances of the capacitor and inductor as ω moves above and below ω_n? What is the value of the transfer function at this frequency ω_n?

ω_n, s.t. Z_L cancels out Z_C

$$X(\omega) = \omega L - \frac{1}{\omega C} = 0$$

$$\left(\omega L = \frac{1}{\omega C}\right) \frac{\omega}{L}$$

$$\omega^2 = \frac{1}{LC} \Rightarrow \boxed{\omega_n = \sqrt{\frac{1}{LC}}}$$

$$H(\omega_n) = \frac{R}{R + j\left(\omega_n L - \frac{1}{\omega_n C}\right)}$$

$$= \frac{R}{R + j\left(L\sqrt{\frac{1}{LC}} - \frac{1}{C\sqrt{\frac{1}{LC}}}\left(\frac{\sqrt{\frac{1}{LC}}}{\sqrt{\frac{1}{LC}}}\right)\right)}$$

$$= \frac{R}{R + j\left(L\sqrt{\frac{1}{LC}} - \frac{\sqrt{\frac{1}{LC}}}{\frac{1}{L}}\right)}$$

$$= \frac{R}{R + j\left(\cancel{L\sqrt{\frac{1}{LC}}} - \cancel{L\sqrt{\frac{1}{LC}}}\right)}$$

$$= \frac{R}{R + 0j} = \frac{R}{R} = 1.$$

R stays the same

X goes to 0

$$X = \omega L - \frac{1}{\omega C}$$

$\omega \ll \omega_n$: $(\omega \to 0)$

$|\omega L| \downarrow \quad \left|-\frac{1}{\omega C}\right| \uparrow$

$$X \cong -\frac{1}{\omega C}$$

$\omega \gg \omega_n$: $(\omega \to \infty)$

$|\omega L| \uparrow \quad \left|-\frac{1}{\omega C}\right| \downarrow$

$$X \cong \omega L$$

(e) **Simplify $X(\omega)$ in two cases, when $\omega \to \infty$ and when $\omega \to 0$. Plug this simplified $X(\omega)$ into your previously solved expressions to find the transfer function at high and low frequencies.**

$X(w) = wL - \frac{1}{wC}$, $H(w) = \frac{R}{R + j\left(wL - \frac{1}{wC}\right)}$

$w \to \infty$: $X(w) \cong wL$

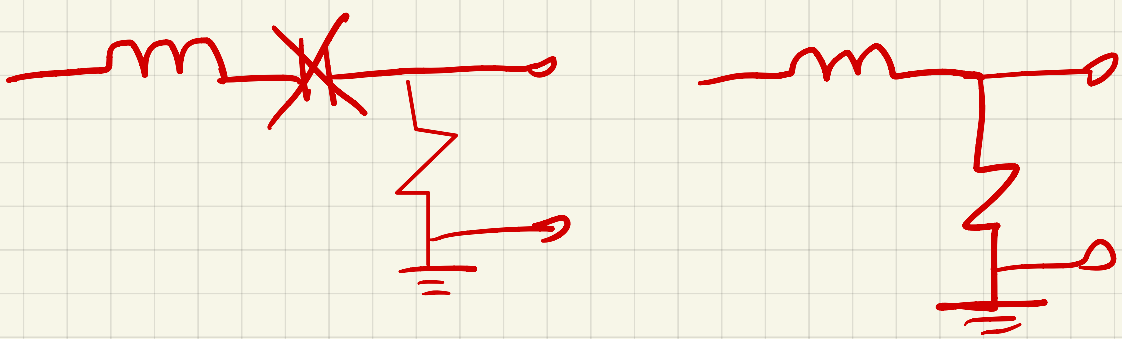
$H(w) = \frac{R}{R + j(wL)} = \frac{R}{R + jwL} = \frac{1}{1 + jw\frac{L}{R}}$

$w_c = \frac{R}{L}$

① $w = 0$: $\frac{1}{1} = 1$

$w = \infty$: $\frac{1}{\infty} = 0$ = Low Pass

②



LR → Low Pass

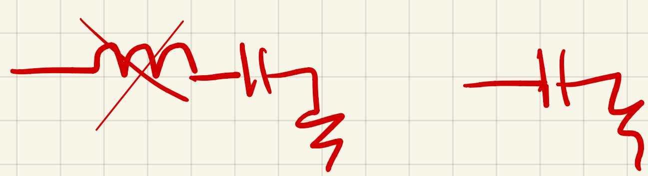
$\frac{1}{1 + \frac{jw}{w_c}}$ $\frac{1}{1 + jwL\left(\frac{1}{\frac{R}{L}}\right)}$

$w \to 0$: $X(w) \cong -\frac{1}{wC}$

$H(w) = \frac{R}{R - j\frac{1}{wC}} = \frac{1}{1 - j\frac{1}{RwC}}$ = High Pass =

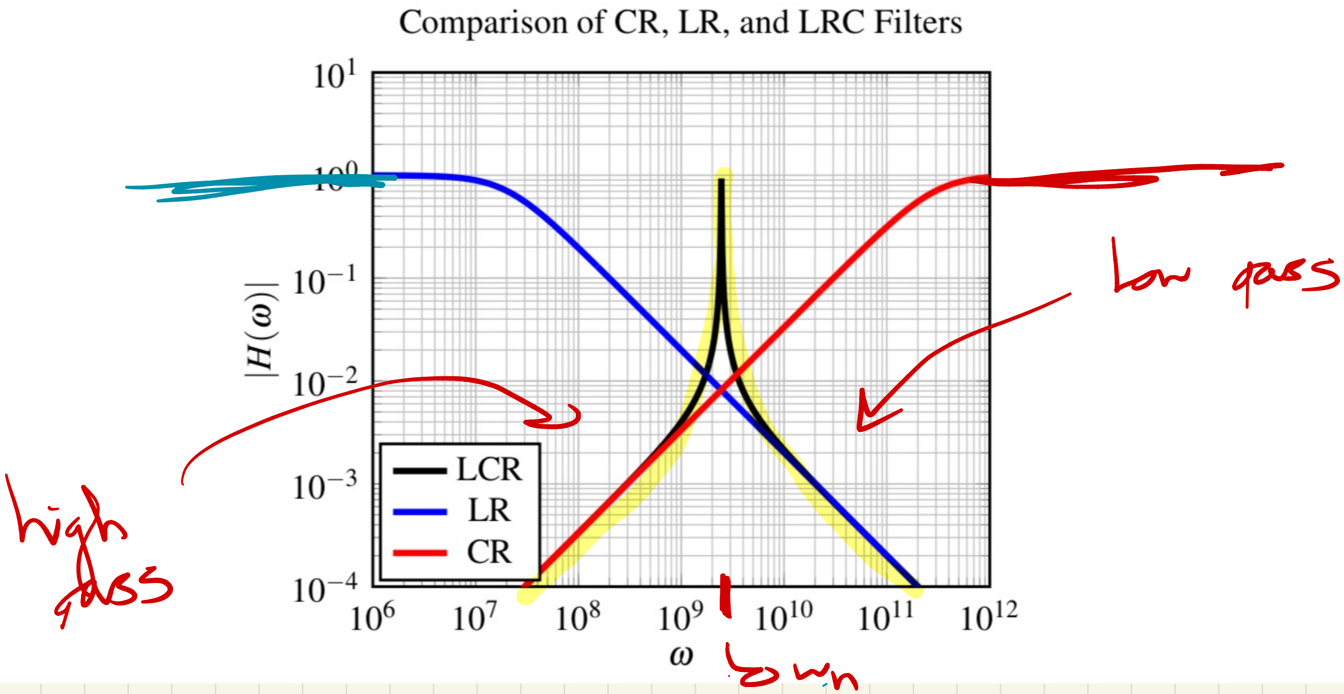
$w = 0$: $\frac{1}{\infty} = 0$

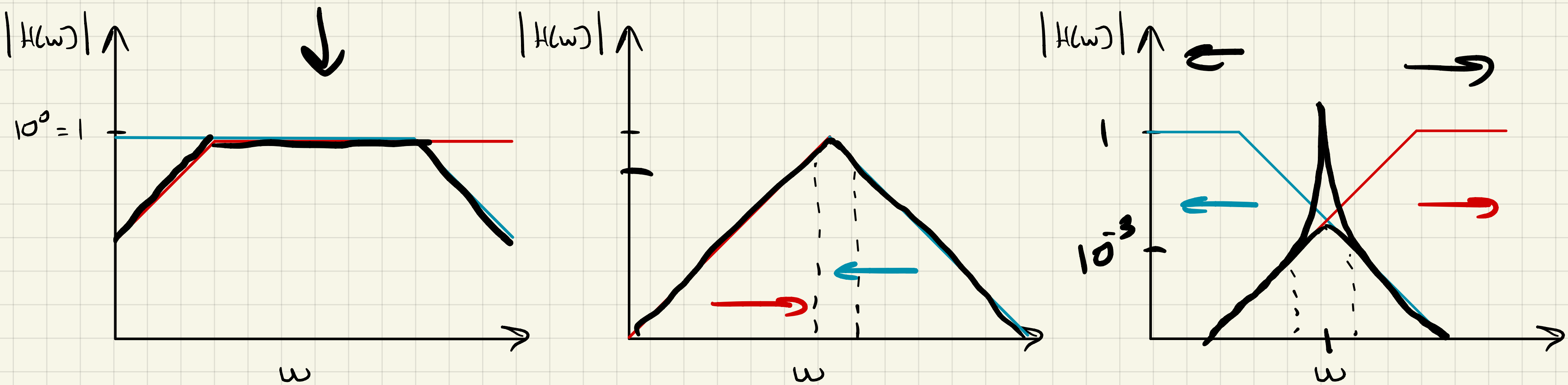
$w = \infty$: $\frac{1}{1} = 1$



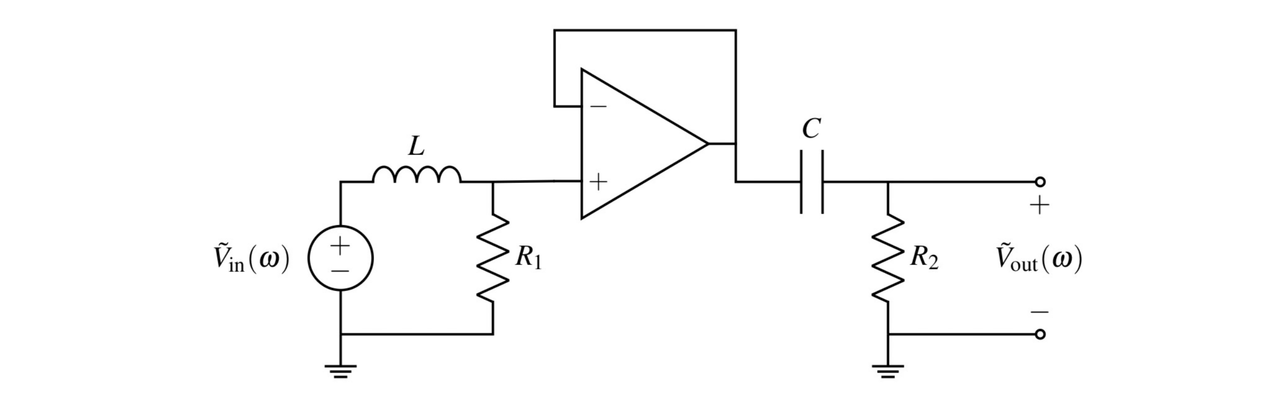
CR → High Pass

Our band-pass filter looks like an LR low-pass filter at high frequencies and a CR high-pass filter at low frequencies. Note that in this case, the cutoff frequencies for the LR and CR filters are not the same as LCR cutoff frequencies or the resonance frequency. In the vicinity of the resonance frequency, notice that the filter is much sharper than any first-order LR or RC filter.





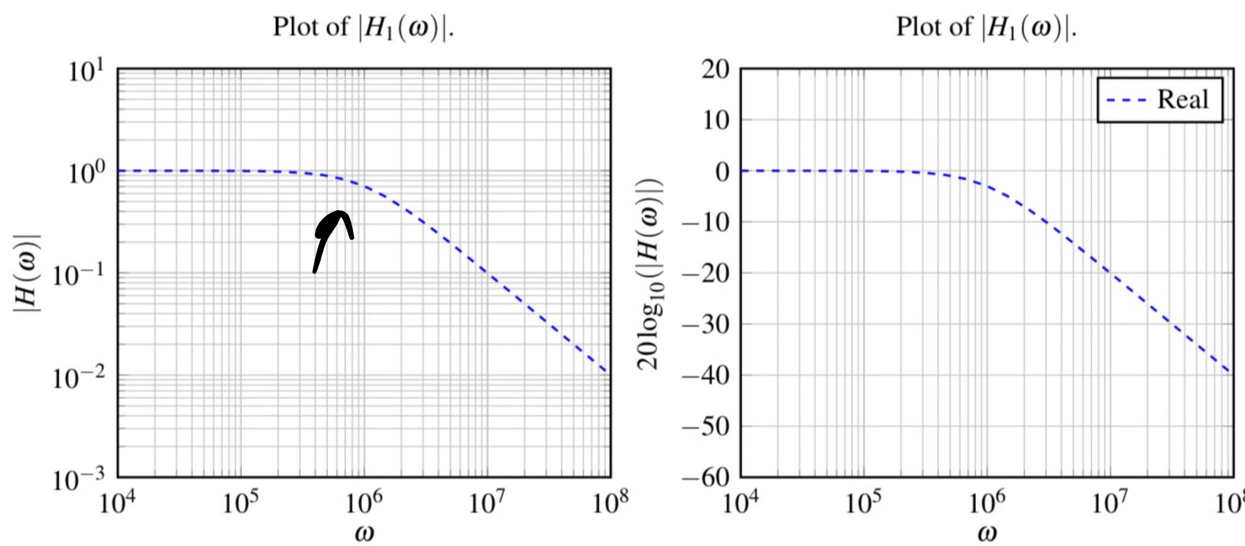
"Regular" Bandpass Filters



Before we start diving into the problem, let's consider a modification of the *magnitude* plot that will help us work with multiple magnitude plots at once. Namely, instead of plotting $|H(\omega)|$ vs. ω where the y-axis is on a *logarithmic* scale, we plot $20\log_{10}(|H(\omega)|)$ vs. ω instead, and now the y-axis is on a *linear* scale.

decibels

The reason that we do this is that, when combining transfer functions, we end up multiplying them. But we really want to add two plots graphically, not multiply them, so we will just plot and add the logarithms. (The constant multiple 20 is there for convention reasons.) Here's what this looks like, with the old grid on the left, and the new grid on the right:



Properties of Logarithms

$$\log_a b = x \iff a^x = b$$

$$\log_{10} 10^3 = 3$$

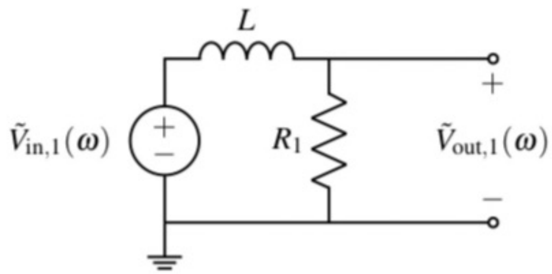
$$\log_2 16 = 4$$

$$\log_a (b \cdot c) = \log_a b + \log_a c$$

[division " = " subtraction]

$$\log_2 64 = \log_2 16 + \log_2 4 = 4 + 2 = 6$$

$$\therefore 20\log_{10}(|H_1 H_2|) = 20\log_{10}(|H_1|) + 20\log_{10}(|H_2|)$$



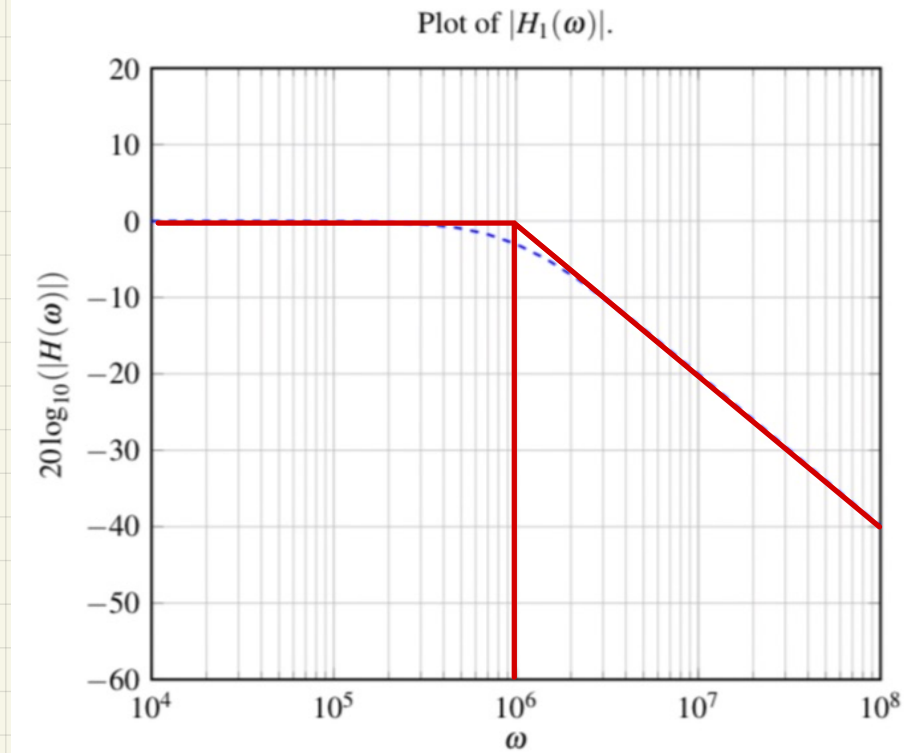
We learned in the previous discussion that the transfer function is given by

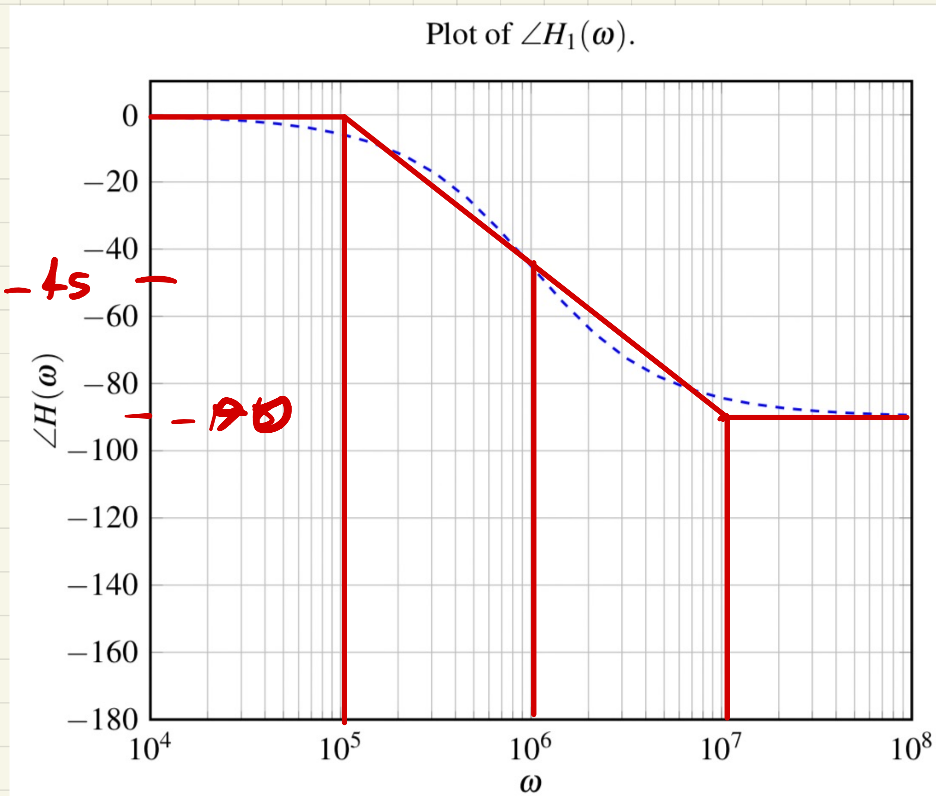
$$H_1(\omega) = \frac{\tilde{V}_{\mathrm{out},1}}{\tilde{V}_{\mathrm{in},1}} = \frac{1}{1 + j\omega\frac{L}{R_1}},$$

the cutoff frequency $\omega_{c,1}$ is given by

$$\omega_{c,1} = \frac{R_1}{L} = \frac{100\,\Omega}{100\,\mu\mathrm{H}} = 1 \times 10^6\,\frac{\mathrm{rad}}{\mathrm{s}},$$

and plots of the transfer function are given by





Magnitude: w_c = "elbow"

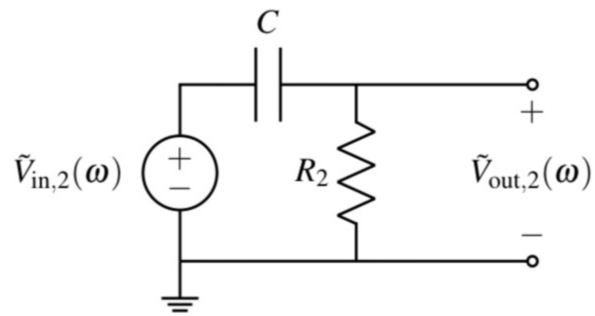
Phase:

$-45 \Longleftrightarrow w_c$

"elbows": $0.1\,w_c$

$10\,w_c$

(b) Consider the second half of the circuit:



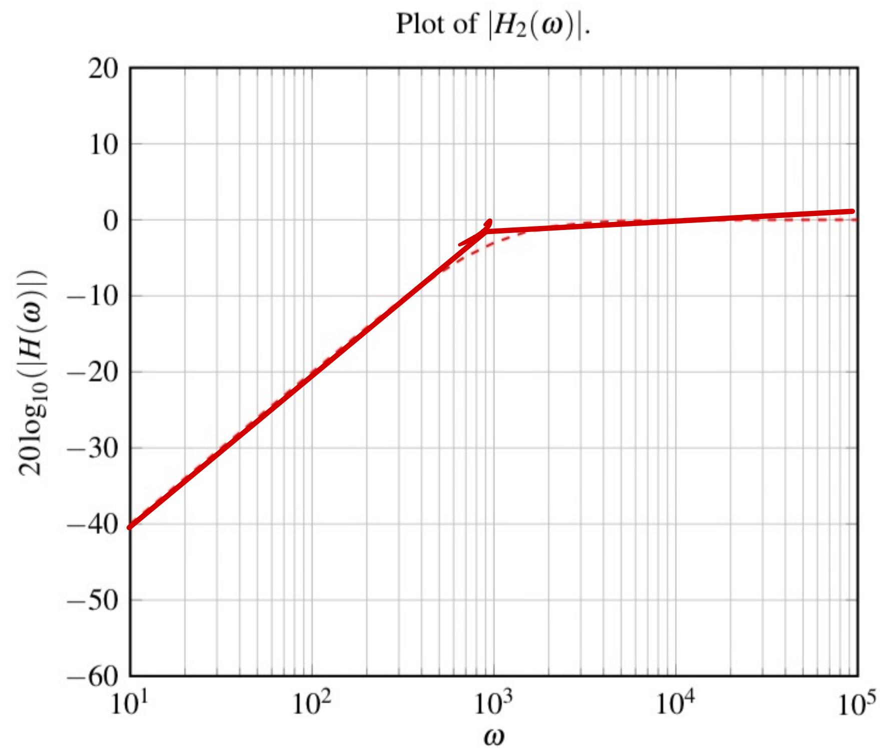
We learned in the previous discussion that the transfer function is given by

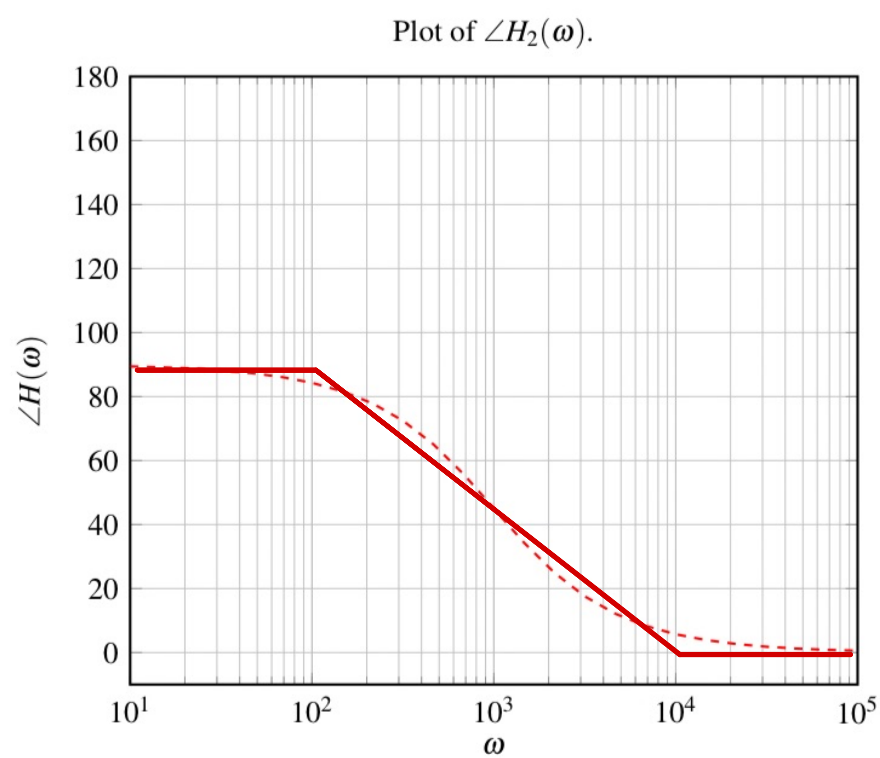
$$H_2(\omega) = \frac{\widetilde{V}_{\text{out},2}}{\widetilde{V}_{\text{in},2}} = \frac{j\omega R_2 C}{1 + j\omega R_2 C},$$

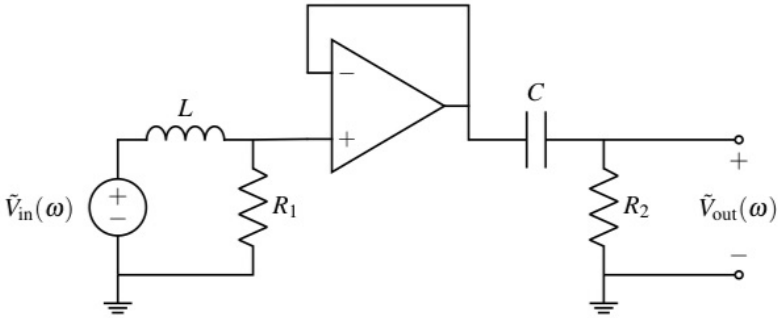
the cutoff frequency $\omega_{c,2}$ is given by

$$\omega_{c,2} = \frac{1}{R_2 C} = \frac{1}{(1\,\text{k}\Omega)\cdot(1\,\mu\text{F})} = 1 \times 10^3\,\frac{\text{rad}}{\text{s}},$$

and plots of the transfer function are given by





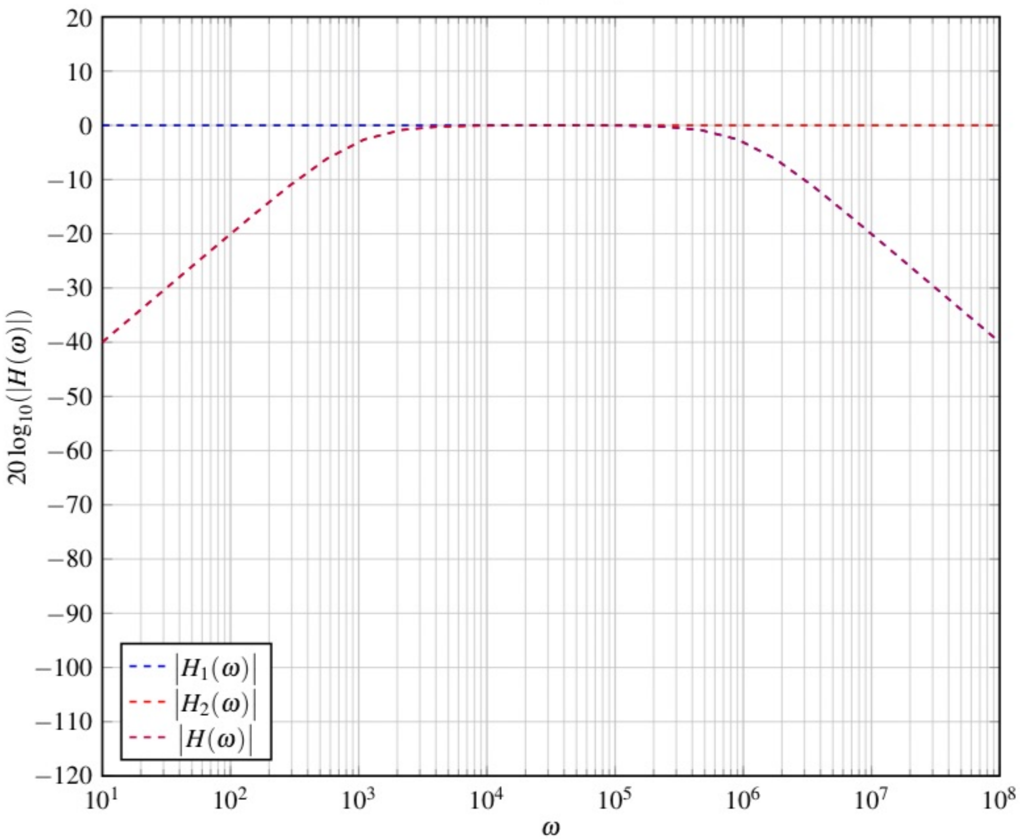


We learned in the previous discussion that the transfer function is

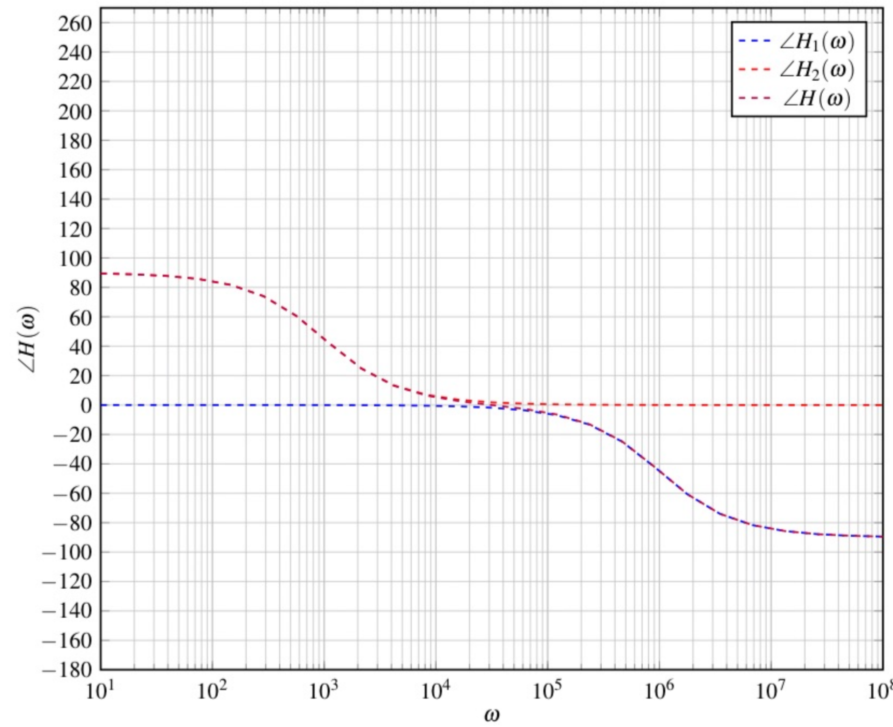
$$H(\omega) = \frac{\widetilde{V}_{\text{out}}}{\widetilde{V}_{\text{in}}} = H_1(\omega)H_2(\omega)$$

and the transfer function plots are given by

Plot of $|H(\omega)|$.



Plot of $\angle H(\omega)$.



On these grids, **draw the Bode plots for magnitude and phase**.

Hint: Recall that

$$\begin{aligned} 20\log_{10}(|H(\omega)|) &= 20\log_{10}(|H_1(\omega)H_2(\omega)|) = 20\log_{10}(|H_1(\omega)||H_2(\omega)|) \\ &= 20\log_{10}(|H_1(\omega)|) + 20\log_{10}(|H_2(\omega)|) \\ \text{and } \angle H(\omega) &= \angle H_1(\omega) + \angle H_2(\omega). \end{aligned}$$

Module 1

Key motivator: Time and Dynamics

Modern electronics and computers are built on Transistors

↳ "Switches", but more complicated

1. Power Consumption (R)
2. Time delay (C)

Connections to 16A (inductors, capacitors)

Transient Analysis + Exponentials & Diff. Eq.

Annoying to Solve – try making problem easier

Systems of Diff. Eq. and Change of Basis

↳ Module 2

Phasors

↓

Transfer fn. + Filters

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